Math 245B Lecture 1 Notes

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1 Forms of The Axiom of Choice

1.1 The axiom of choice

We will need the axiom of choice later, so we will begin the course by introducing it now.

Definition 1.1. Let A be a nonempty set, and let X_α be a set for each $\alpha \in A$. The **Cartesian product** $\prod_{\alpha \in A} X_\alpha = \{\langle x_\alpha \rangle_{\alpha \in A} : a_\alpha \in X_\alpha \forall \alpha \in A\}$ is a function $A \to \bigcup_{\alpha \in A} X_\alpha$ such that $\alpha \mapsto x_\alpha$..

Definition 1.2. The Axiom of choice says that if $X_\alpha \neq \varnothing$ then $\prod_{\alpha \in A} X_\alpha \neq \varnothing$.

Theorem 1.1 (Cohen)**.** *The axiom of choice is not implied by the other standard axioms of set theory.*

This is difficult to apply, but we will use provably equivalent statements.

1.2 Posets and Zorn's lemma

Let X be a set.

Definition 1.3. A **partial order** on X is a relation "$\leq$" on X that is

1. Transitive: if $a \leq y$ and $y \leq z$, then $x \leq z$
2. Reflexive $x \leq x$ for all $x \in X$
3. Anti-symmetric: if $x \leq y$ and $y \leq x$, then $x = y$

Definition 1.4. A **total order** is a partial order where for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Example 1.1. Let S be a set and let $\mathscr{P}(S)$ be the set of subsets of S¿ Then $\subseteq$ is a partial order on $\mathscr{P}(S)$.

Example 1.2. On $\mathbb{R}$, $\leq$ is a partial order (and in fact a total order).

Example 1.3. Let $U \subseteq \mathbb{R}^2$ be a domain. Say $(x_1, y_1) \leq (x_2, x_2)$ if $y_2 \geq y_1$ and $|y_2 - y_1| \leq |x_2 - x_1|$. This is a partial order but not a total order.

Definition 1.5. Let $(X, \leq)$ be a poset with $U \subseteq X$. An element $x \in U$ is **maximal** if when $y \in U$ and $y \geq x$, we must have $y = x$. An element $x \in X$ is an **upper bound** for U if $x \geq u$ for all $u \in U$.

The definitions of minimal elements and lower bounds are analogous.

Definition 1.6. A **chain** in a partially ordered set $(X, \leq)$ is a subset $Y \subseteq X$ such that for all $y, z \in Y$, either $y \leq z$ or $z \leq y$.

Theorem 1.2 (Hausdorff Maximal Principal)**.** *Any nonempty poset $(X, \leq)$ has a maximal chain $Y \subseteq X$.*

Lemma 1.1 (Zorn)**.** *Let $(X, \leq)$ be a nonempty poset. If every chain in X has an upper bound,, then X has a maimal element.*

1.3 Proof sketch of Zorn's lemma and the Hausdorff maximality principle

Here is another incarnation of the axiom of choice.

Theorem 1.3. *Let $S \neq \varnothing$, and let $\mathcal{F} \subseteq \mathscr{P}(S)$ with $\mathcal{F} \neq \varnothing$. Assume $\mathcal{F}$ is*

1. *down-closed: If $A \subseteq B \in \mathcal{F}$, then $A \in \mathcal{F}$*
2. *chain-closed: If C is a chain with $C \subseteq \mathcal{F}$, then $\bigcup C \in \mathcal{F}$.*

Then $\mathcal{F}$ contains a maximal element.

Here is a sketch of the proof.

Proof. First, $\varnothing \in \mathcal{F}$, so $\mathcal{F} \neq \varnothing$. Assume the result is false. THen for all $A \in \mathcal{F}$, there exists a nonempty $B \in S \setminus A$ such that $A \cup B \in \mathcal{F}$. By property 1, we may assume $|B| = 1$. By the axiom of choice, there exists $f : \mathcal{F} \to S$ such that $f(A) \in S \setminus A$ and $A \cup \{f(A)\} \in \mathcal{F}$.

At this point, the idea is to start at the empty set and keep constructing chains, then taking the union of the chain, and then continuing. This requires a notion of the well-ordering principle, so we will choose a different explanation for our sketch.

Call a subfamily $T \subseteq \mathcal{F}$ a **tower** if

1. $\varnothing \in T$
2. $A \in T \implies A \cup \{f(A)\} \in T$
3. T is chain-closed.

Towers exist (e.g. $\mathcal{F}$). Any intersections of towers is a tower. So there exists a minimal tower $T_{\min}$.

Call $A \in T_{\min}$ a **bottleneck**[1] if $\forall B \in T_{\min}$, either $A \subseteq B$ or $B \subseteq A$. The idea is that the set of bottlenecks is a tower. So $T_{\min}$ is a chain. By property 3, $\bigcup T_{\min} \in T_{\min}$. So by property 2, $\bigcup T \cup \{f(\bigcup T_{\min})\} \in T_{\min}$. This is impossible. $\square$

Here is how we prove the Hausdorff maximal principle:

Proof. Let $(X, \leq)$ be nonempty. Let $\mathcal{F}$ be the set of chains in X. This satisfies the conditions of the theorem, which implies that there exists a maximal chain. $\square$

We can prove Zorn's lemma from this, as well.

Proof. Take an upper bound for a maximal chain Such an element is maximal. $\square$

[1] This is not standard notation.